Aharonov-Bohm effect on the Poincaré disk

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Abstract

We consider formal quantum Hamiltonian of a charged particle on the Poincaré disk in the presence of an Aharonov-Bohm magnetic vortex and a uniform magnetic field. It is shown that this Hamiltonian admits a four-parameter family of self-adjoint extensions. Its resolvent and the density of states are calculated for natural values of the extension parameters.

1 Introduction

Quantum dynamics on the Poincaré disk has long been a subject of theoretical interest, mainly because of the insights its study provides into the theory of quantum chaos. Analyzed examples include, for instance, the free motion under the action of constant magnetic fields [8, 9, 23], the Kepler problem [25], the scattering by the Aharonov-Bohm (AB) [24, 26] and Aharonov-Bohm-Coulomb [29] potentials, the study of point interactions [3, 5] and quantum Hall effect [6].

In the present paper, we consider the Hamiltonian of a charged spinless particle moving on the hyperbolic disk, pierced by an AB flux, in the presence of a uniform magnetic field. First part of this work is rather standard: we determine the admissible boundary conditions on the wave functions, using Krein's theory of self-adjoint extensions (SAEs) [4]. It turns out that in the most general case the formal Hamiltonian has deficiency indices $(2, 2)$ and thus admits a four-parameter family of SAEs. Let us remark that similar results on the plane have been found in [2, 11] in the case of zero magnetic field, and in [21] for non-zero fields; SAEs of the Dirac Hamiltonian on the plane have been studied in [16, 22].

The rest of this paper is devoted to the study of a particular extension, corresponding to the choice of regular boundary conditions at the position of the AB flux. We start by constructing certain integral representations for common eigenstates of this Hamiltonian and the angular momentum operator. These representations then allow to sum up the contributions coming from different angular momenta to the resolvent kernel, and to evaluate this kernel and the density of states in a closed form.

The material is organized as follows. In Section 2, we introduce basic notations and study elementary solutions of the radial Schrödinger equation on the Poincaré disk. Self-adjointness of the full AB Hamiltonian is discussed in Section 3. In Section 4 we find a compact expression for the resolvent of the regular extension (formulas 4.14), (4.19)–(4.22)). These relations represent the main result of the present work. The density of states, induced by the AB flux in the whole hyperbolic space (see (5.8)–(5.10)), is obtained in Section 5. Some technical results are relegated to the appendices.

2 Free Hamiltonian on the Poincaré disk

2.1 Basic formulas

Let us identify the Poincaré disk $D = SU(1,1)/SO(2)$ with the interior of the unit circle $|z|^2 < 1$ in the complex plane, equipped with the metric

$$ds^2 = g_{z\bar{z}}\, dz\, d\bar{z} = R^2 \frac{dz\, d\bar{z}}{\left(1-|z|^2\right)^2} \tag{2.1}$$

of constant Gaussian curvature $-4/R^2$. We consider a spinless particle moving on the disk and interacting with a magnetic field. The latter can be introduced as a connection 1-form

$$\mathcal{A} = A_z\, dz + A_{\bar{z}}\, d\bar{z}$$

on the trivial $U(1)$-bundle over D. Quantum dynamics of a particle of unit charge is described by the Hamiltonian

$$\hat{H} = -\frac{2}{g_{z\bar{z}}}\, \{D_z, D_{\bar{z}}\}\,, \tag{2.2}$$

where $D_z = \partial_z + iA_z$ and $D_{\bar{z}} = \partial_{\bar{z}} + iA_{\bar{z}}$ are the usual covariant derivatives. To unburden formulas, we put the particle mass equal to $1/2$ and $\hbar = c = 1$ throughout the paper.

In the remainder of the present section, the following vector potential is considered:

$$\mathcal{A}^{(B)} = -\frac{iBR^2}{4}\, \frac{\bar{z}\, dz - z\, d\bar{z}}{1-|z|^2}\,. \tag{2.3}$$

It generates a curvature 2-form $\mathcal{F}^{(B)}$, proportional to the invariant volume measure $d\mu = \frac{i}{2}\, g_{z\bar{z}}\, dz \wedge d\bar{z}$. Indeed, we have

$$\mathcal{F}^{(B)} = d\mathcal{A}^{(B)} = B d\mu\,.$$

Therefore, the potential (2.3) describes a uniform magnetic field of intensity B. Introducing polar coordinates $z = re^{i\varphi}$, $\bar{z} = re^{-i\varphi}$, one can write the corresponding Hamiltonian as

$$\hat{H}^{(B)} = -\frac{\left(1-r^2\right)^2}{R^2} \left\{ \partial_{rr} + \frac{1}{r}\, \partial_r + \frac{1}{r^2}\, \partial_{\varphi\varphi} + \frac{iBR^2}{1-r^2}\, \partial_\varphi - \frac{B^2R^4}{4\left(1-r^2\right)^2}\, r^2 \right\}. \tag{2.4}$$

Note that the domain of $\hat{H}^{(B)}$ is not yet specified. It will be fixed in the next section by the requirement for the Hamiltonian to be a self-adjoint operator. According to Stone's theorem, this condition ensures the existence of consistent dynamics.

2.2 Radial Hamiltonians

Formal Hamiltonian $\hat{H}^{(B)}$ commutes with the angular momentum operator $\hat{L} = -i\partial_\varphi$. Therefore, it leaves invariant the eigenspaces of $\hat{L}$, spanned by the functions $w_l(r)e^{il\varphi}$ ($l \in \mathbb{Z}$). Being restricted to the eigenspace of $\hat{L}$, characterized by the angular momentum l, the Hamiltonian acts as follows:

$$w_l(r) \mapsto \hat{H}_l\, w_l(r),$$

$$\hat{H}_l = -\frac{(1-r^2)^2}{R^2}\left\{\partial_{rr} + \frac{1}{r}\,\partial_r - \frac{l^2}{r^2} - \frac{4b\,l}{1-r^2} - \frac{4b^2r^2}{(1-r^2)^2}\right\}. \tag{2.5}$$

Here we have introduced instead of B a dimensionless parameter $b = BR^2/4$.

It will be useful for us to let the parameter l to take on not only integer, but also arbitrary real values, and to study in some detail the properties of solutions of the radial Schrödinger equation

$$\left(\hat{H}_l - k^2\right) w_l = 0. \tag{2.6}$$

In what follows it will be always assumed that $k^2 \in \mathbb{C}\backslash\mathbb{R}^+ \cup \{0\}$. It is also convenient to introduce instead of r a new variable $t = r^2$.

We are interested in the solutions of (2.6) leading to square integrable (with the measure $d\mu$) functions on D. These solutions should be then square integrable on the open interval $I = (0,1)$ with the measure $d\mu_t = \dfrac{R^2 dt}{2(1-t)^2}$. For each $l \in \mathbb{R}$ there exists only one solution of (2.6), which is square integrable in the neighbourhood of the point $t = 1$. Its explicit form is

$$\begin{aligned} w_l^{(I)}(t) &= t^{-l/2}\,(1-t)^{\chi}\,{}_2F_1\left(\chi - b, \chi + b - l, 2\chi, 1-t\right) = \\ &= t^{l/2}\;\;(1-t)^{\chi}\,{}_2F_1\left(\chi + b, \chi - b + l, 2\chi, 1-t\right), \end{aligned} \tag{2.7}$$

where

$$\chi = \frac{1+\sqrt{1+4b^2-k^2R^2}}{2}$$

and ${}_2F_1(\alpha,\beta,\gamma,z)$ denotes Gauss hypergeometric function. The branches of square roots are defined so that they take on real positive values for purely imaginary k.

Similarly, for each $l \in (-\infty,-1] \cup [1,\infty)$ there is only one solution of (2.6), which is square integrable with respect to $d\mu_t$ near the point $t = 0$. The form of this solution depends on whether $l \geq 1$ or $l \leq -1$. In the first case, i. e. for $l \geq 1$, it is given by

$$w_l^{(II,+)}(t) = t^{l/2}\,(1-t)^{\chi}\,{}_2F_1\left(\chi + b, \chi - b + l, 1 + l, t\right) \tag{2.8}$$

while for $l \leq -1$ this solution is written as follows:

$$w_l^{(II,-)}(t) = t^{-l/2}\,(1-t)^{\chi}\,{}_2F_1\left(\chi - b, \chi + b - l, 1 - l, t\right). \tag{2.9}$$

Note that for $|l| < 1$ both functions $w_l^{(II,\pm)}(t)$ are square integrable in the vicinity of the point $t = 0$ and solve the radial Schrödinger equation (2.6). These solutions are linearly

independent except for $l = 0$. However, in the latter case the equation (2.6) still admits two distinct solutions that are square integrable as $t \to 0$:

$$w_0^{(II)}(t) = (1-t)^{\chi}\, u(t), \qquad \tilde{w}_0^{(II)}(t) = (1-t)^{\chi}\, v(t),$$

where u and v are any two linearly independent solutions of the hypergeometric equation with parameters $\alpha = \chi + b$, $\beta = \chi - b$, $\gamma = 1$ (one can choose them, for instance, according to the formulas 15.5.16 and 15.5.17 of [1]).

Let us now show that the solutions $w_l^{(I)}(t)$ and $w_l^{(II,+)}(t)$ are linearly independent for $l > -1$, and the solutions $w_l^{(I)}(t)$ and $w_l^{(II,-)}(t)$ are linearly independent for $l < 1$. This can be done by an explicit computation of their Wronskian

$$W(f_1, f_2) = f_1 \cdot \partial_t f_2 - \partial_t f_1 \cdot f_2\,.$$

Namely, using the connection and analytic continuation formulas for hypergeometric functions [1], one obtains

$$W\left(w_l^{(I)}(t), w_l^{(II,\pm)}(t)\right) = \left(t\, C_{k,l}^{\pm}\right)^{-1}, \tag{2.10}$$

with

$$C_{k,l}^{\pm} = \frac{\Gamma(\chi \pm b)\Gamma(\chi \mp b \pm l)}{\Gamma(2\chi)\Gamma(1 \pm l)}\,. \tag{2.11}$$

Therefore, for $k^2 \in \mathbb{C}\backslash\mathbb{R}^+ \cup \{0\}$ and $|l| \geq 1$ the equation (2.6) has no square integrable solutions (with the measure $d\mu_t$) on the whole interval I. This is true, in particular, for all radial Hamiltonians $\hat{H}_{l\in\mathbb{Z}}$ of the free particle in a uniform magnetic field, except for the s-wave Hamiltonian $\hat{H}_0$. In the case $|l| < 1$ the equation (2.6) has exactly one square integrable solution, given by the formula (2.7).

Let us now restrict the domain of $\hat{H}_l$ to $\mathcal{D}(\hat{H}_l) = C_0^{\infty}(I)$, i. e. to smooth compactly supported functions. Then the above remarks imply that

- $\hat{H}_l$ is essentially self-adjoint for $|l| \geq 1$,
- for $|l| < 1$ the operator $\hat{H}_l$ has deficiency indices $(1, 1)$ and thus admits a one-parameter family of self-adjoint extensions (SAEs).

Different extensions $\hat{H}_l^{(\gamma)}$ ($|l| < 1$) are in one-to-one correspondence with the isometries between the deficiency subspaces $\mathcal{K}_l^{\pm} = \ker\left(\hat{H}_l \mp i\varepsilon\right)$, where $\varepsilon \in \mathbb{R}^+$ may be chosen arbitrarily. They can be labeled by a real parameter $\gamma \in [0, 2\pi)$ and characterized by the domains

$$\mathcal{D}(\hat{H}_l^{(\gamma)}) = \left\{ f + c\left(w_l^+ + e^{i\gamma} w_l^-\right) \mid f \in C_0^{\infty}(I), c \in \mathbb{C} \right\},$$

where the functions $w_l^{\pm}(t)$ may be chosen as follows:

$$w_l^{\pm}(t) = w_l^{(I)}(t)\Big|_{k^2 = \pm i\varepsilon}. \tag{2.12}$$

Remark. For a particular value of γ the domain $\mathcal{D}(\hat{H}_l^{(\gamma)})$ is composed of functions, regular at $t = 0$. The corresponding SAE of $\hat{H}_l$ will be denoted by $\hat{H}_l^{\mathrm{reg}}$.

2.3 Resolvent

The kernel $G_{k,l}(t,t')$ of the resolvent of the radial Hamiltonian $\hat{H}_l$ satisfies the equation

$$\left(\hat{H}_l(t)-k^2\right)G_{k,l}(t,t')=\frac{2(1-t)^2}{R^2}\,\delta(t-t')\,. \tag{2.13}$$

It basically means that if $\left(\hat{H}_l-k^2\right)u=v$ for some $u\in\mathcal{D}(\hat{H}_l)$, then

$$u(t)=\int_I G_{k,l}(t,t')v(t')\,d\mu_{t'}.$$

In order to find the solution of the equation (2.13), consider the following ansatz:

$$G_{k,l}(t,t')=\begin{cases}\tilde{C}^{\pm}_{k,l}\;w_l^{(II,\pm)}(t)\;w_l^{(I)}(t') & \text{for}\;\;0<t<t'<1,\\ \tilde{C}^{\pm}_{k,l}\;w_l^{(I)}(t)\;w_l^{(II,\pm)}(t') & \text{for}\;\;0<t'<t<1,\end{cases} \tag{2.14}$$

where the signs "+" and "−" should be chosen for $l\geq 0$ and $l<0$, correspondingly. It is clear that the function, defined by (2.14), solves the equation (2.13) for $t\neq t'$ and satisfies the boundary conditions of square integrability at the points $t=0$ and $t=1$. (In the case $|l|<1$ the requirement of square integrability at the boundary points is not sufficient to make the operator $\hat{H}_l$ self-adjoint; however, for such l, the ansatz (2.14) also satisfies the regularity condition at $t=0$ and thus corresponds to the resolvent of the extension $\hat{H}_l^{\mathrm{reg}}$).

Taking into account the explicit form of the operator $\hat{H}_l$, one may show that the required singular behaviour of the Green function at the point $t=t'$ is guaranteed provided the condition

$$\partial_{t'}G_{k,l}(t',t)\Big|_{t-0}^{t+0}=-\frac{1}{2t}$$

holds. Using (2.14), one can rewrite this condition as

$$2t\,\tilde{C}^{\pm}_{k,l}\cdot W\left(w_l^{(I)}(t),w_l^{(II,\pm)}(t)\right)=1\,.$$

It follows from (2.10) that the last relation is satisfied if we choose $\tilde{C}^{\pm}_{k,l}=C^{\pm}_{k,l}/2$. Substituting this expression into (2.14), one finds a representation for the radial Green functions $G_{k,l}(t,t')$.

3 Hamiltonian in the presence of a magnetic vortex

3.1 Radial Hamiltonians

Let us now add to the Hamiltonian the field of an Aharonov-Bohm magnetic flux $\Phi=2\pi\nu$, centered at $z=0$:

$$\mathcal{A}^{(v)}=-\frac{i\nu}{2}\left(\frac{dz}{z}-\frac{d\bar{z}}{\bar{z}}\right). \tag{3.1}$$

This choice of the flux position involves no loss of generality, since we have a well-known transitive $SU(1,1)$-action on D, which preserves the metric (2.1):

$$z \mapsto z_g = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}, \qquad g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(1,1). \tag{3.2}$$

Any gauge field configuration corresponding to a single vortex and a uniform magnetic field can be reduced to $\mathcal{A} = \mathcal{A}^{(B)} + \mathcal{A}^{(v)}$, using the transformation (3.2) combined with a gauge change.

The Hamiltonian (2.2) in the presence of a vortex has thus the following form:

$$\hat{H}_v = -\frac{(1-r^2)^2}{R^2} \left\{ \partial_{rr} + \frac{1}{r}\,\partial_r + \frac{1}{r^2}\left(\partial_\varphi + i\nu\right)^2 + \frac{4ib}{1-r^2}\left(\partial_\varphi + i\nu\right) - \frac{4b^2}{(1-r^2)^2}\, r^2 \right\}. \tag{3.3}$$

This Hamiltonian still commutes with the angular momentum operator $\hat{L}$. Radial Hamiltonians $\hat{H}_{v,l}$ are obtained by the restriction of $\hat{H}_v$ to the eigenspaces of $\hat{L}$ with fixed angular momenta $l \in \mathbb{Z}$. Namely, one obtains $\hat{H}_{v,l} = \hat{H}_{l+\nu}$, where the operators $\hat{H}_{\alpha\in\mathbb{R}}$ are defined as in (2.5). Thus the only effect the AB vortex has on the formal Hamiltonians is the shift of the angular momentum variable by ν. This observation allows to considerably simplify the derivation of many results, using the calculations from the previous section.

Remark. As usual, for integer flux values some further simplifications occur. The Hamiltonians $\hat{H}^{(B)}$ and $\hat{H}_v$ are related by a gauge transformation

$$\hat{H}_v = U \hat{H}^{(B)} U^\dagger, \qquad U : w \mapsto e^{-i\nu\varphi} w, \tag{3.4}$$

which is globally well-defined for $\nu \in \mathbb{Z}$. The kernels of the resolvents of $\hat{H}^{(B)}$ and $\hat{H}_v$ in this case differ only by a factor of $e^{i\nu(\varphi-\varphi')}$, and this change has no effect on the observable quantities.

3.2 Self-adjointness

From now on it will be assumed that $-1 < \nu \leq 0$ (it is clear from the above that this involves no loss of generality). Let us consider the full Hamiltonian $\hat{H}_v$ and restrict its domain to functions with compact support on the punctured disk: $\mathcal{D}(\hat{H}_v) = C_0^\infty(D\backslash\{0\})$. It was shown in the previous section that for $|l| \geq 1$ the operator $\hat{H}_l$ is essentially self-adjoint, and for $|l| < 1$ it has deficiency indices $(1,1)$. One should then distinguish two cases:

- $\underline{\nu = 0}$. In this case $\hat{H}_v$ has deficiency indices $(1,1)$ and admits a one-parameter family of SAEs $\hat{H}_v^{(\gamma)}$ with $\gamma \in [0, 2\pi)$ and

  $$\mathcal{D}(\hat{H}_v^{(\gamma)}) = \left\{ f + c\left(w_0^+ + e^{i\gamma} w_0^-\right) \mid f \in C_0^\infty(D\backslash\{0\}), c \in \mathbb{C} \right\}.$$

  These Hamiltonians describe a purely contact (non-magnetic) interaction of a particle with the AB solenoid. They have already been considered in [3], so we will not pursue their study.

- $\underline{-1 < \nu < 0}$. For such ν the deficiency subspaces $\mathcal{K}^{\pm}$ of the full Hamiltonian $\hat{H}_v$ are generated by those of the operators $\hat{H}_\nu$ and $\hat{H}_{1+\nu}$. Thus $\hat{H}_v$ has deficiency indices $(2,2)$ and admits a four-parameter family of SAEs. Different extensions can be labeled by a unitary 2×2 matrix U and characterized by the domains

$$\mathcal{D}(\hat{H}_v^U) = \left\{ f + \sum_{i=1,2} c_i \Big(\mathrm{w}_i^+ + \sum_{j=1,2} U_{ij}\, \mathrm{w}_j^- \Big) \,\Big|\, f \in C_0^\infty(D\backslash\{0\}), c_{1,2} \in \mathbb{C} \right\}.$$

where $\mathrm{w}_{1,2}^{\pm}$ are orthonormal elements of the bases of $\mathcal{K}^{\pm}$,

$$\mathrm{w}_1^{\pm}(t,\varphi) = \frac{w_\nu^{\pm}(t)}{\|w_\nu^{\pm}(t)\|}, \qquad \mathrm{w}_2^{\pm}(t,\varphi) = \frac{w_{1+\nu}^{\pm}(t)}{\|w_{1+\nu}^{\pm}(t)\|}\, e^{i\varphi},$$

and $\|\cdot\|$ denotes the L^2-norm on I with respect to the measure $d\mu_t$.

Note that the diagonal matrix U describes magnetic point interactions acting separately in s-channel ($l=0$) and p-channel ($l=1$). Non-diagonal U introduces a coupling between the two modes so that the Hamiltonian no longer commutes with the angular momentum.

Further analysis of spectral properties of H_v^U is a bit cumbersome in the general case (see, for example, the papers [21], [2, 11], where such an analysis has been performed for the AB effect on the plane with and without magnetic field). We remark, however, that there exists a distinguished SAE of $\hat{H}_v$, whose domain consists of functions vanishing for $t \to 0$. This extension will be denoted by $\hat{H}_v^{\mathrm{reg}}$. The next section is devoted to the calculation of its resolvent $\Big(\hat{H}_v^{\mathrm{reg}} - k^2\Big)^{-1}$. The resolvent of any other SAE can be obtained from the latter using Krein's formula [4].

4 One-vortex resolvent

4.1 Contour integral representations of the radial waves

The main technical difficulty in the calculation of the resolvent kernel $G_k(z,z')$ of the Hamiltonian $\hat{H}_v^{\mathrm{reg}}$ is the summation of radial contributions coming from different angular momenta:

$$G_k(z,z') = \frac{1}{2\pi} \sum_{l\in\mathbb{Z}} G_{k,l+\nu}(t,t') e^{il(\varphi-\varphi')}. \tag{4.1}$$

In order to address this problem, it is useful to introduce instead of the radial waves (2.7)–(2.9) the functions depending on both t and φ:

$$\mathrm{w}_l^{(I)}(z) = \frac{\Gamma(\chi+b)\Gamma(\chi-b)}{\Gamma(2\chi)}\, e^{\,il(\varphi+\pi)}\, w_l^{(I)}(t), \tag{4.2}$$

$$\hat{\mathrm{w}}_l^{(I)}(z) = \frac{\Gamma(\chi+b)\Gamma(\chi-b)}{\Gamma(2\chi)}\, e^{-il(\varphi+\pi)} w_l^{(I)}(t), \tag{4.3}$$

$$\mathrm{w}_l^{(II,\pm)}(z) = 2\pi i\,\frac{\Gamma(\chi\mp b\pm l)}{\Gamma(\chi\mp b)\Gamma(1\pm l)}\,e^{\,il(\varphi+\pi)}\,w_l^{(II,\pm)}(t), \tag{4.4}$$

$$\hat{\mathrm{w}}_l^{(II,\pm)}(z) = 2\pi i\,\frac{\Gamma(\chi\mp b\pm l)}{\Gamma(\chi\mp b)\Gamma(1\pm l)}\,e^{-il(\varphi+\pi)}w_l^{(II,\pm)}(t). \tag{4.5}$$

Combining these formulas with the relations (2.14), (2.11), one can rewrite the Green function (4.1) in the following way:

$$G_k(z,z') = \frac{e^{-i\nu(\varphi-\varphi')}}{8i\pi^2}\left(\mathcal{G}_k^{(+)}(z,z') + \mathcal{G}_k^{(-)}(z,z')\right), \tag{4.6}$$

where the functions $\mathcal{G}_k^{(\pm)}(z,z')$ are given by

$$\mathcal{G}_k^{(\pm)}(z,z') = \sum_{l\in\mathbb{Z}+\nu,\, l\gtrless 0} \mathrm{w}_l^{(I)}(z)\hat{\mathrm{w}}_l^{(II,\pm)}(z') \qquad \text{for } |z|>|z'|, \tag{4.7}$$

$$\mathcal{G}_k^{(\pm)}(z,z') = \sum_{l\in\mathbb{Z}+\nu,\, l\gtrless 0} \mathrm{w}_l^{(II,\pm)}(z)\hat{\mathrm{w}}_l^{(I)}(z') \qquad \text{for } |z|<|z'|. \tag{4.8}$$

The sums (4.7)–(4.8) can be computed using a special set of solutions of stationary Schrödinger equation without AB flux, known as horocyclic waves [12]. These solutions have the form

$$\Psi_\pm(z,\theta) = \frac{(1-|z|^2)^{\chi_\pm}}{(1+z\,e^{-\theta})^{\chi_\pm-b}\,(1+\bar{z}\,e^{\theta})^{\chi_\pm+b}}\,, \tag{4.9}$$

where

$$\chi_\pm = \frac{1}{2}\pm\left(\chi-\frac{1}{2}\right)$$

and θ is an arbitrary complex parameter. Being considered as functions of θ, horocyclic waves $\Psi_\pm(z,\theta)$ have an infinite number of branchpoints located at $\theta = \pm\ln r + i\,(\varphi+\pi+2\pi\mathbb{Z})$. Let us introduce a system of branch cuts in the θ-plane as shown in the Fig. 1. The sheets of Riemann surfaces of the functions $\Psi_\pm(z,\theta)$ are fixed by the requirement that the arguments of both $1+z\,e^{-\theta}$ and $1+\bar{z}\,e^{\theta}$ are equal to zero on the line $\mathrm{Im}\,\theta=\varphi$.

Recall that the Hamiltonians $\hat{H}^{(B)}$ and $\hat{H}_v$ are related by the gauge transformation (3.4). Although this transformation is singular for non-integer values of the flux, one can still relate any solution of the equation $(\hat{H}_v-k^2)w=0$ to a solution of the same equation without AB field, $(\hat{H}^{(B)}-k^2)\psi=0$. However, since we have $w=e^{-i\nu\varphi}\psi$, the function ψ should be branched with the monodromy $e^{2\pi i\nu}$ at the point $z=0$. Motivated by this well-known fact, we will try to represent radial wave functions (4.2)–(4.5) as superpositions of elementary solutions (4.9),

$$\mathrm{w}(z) = \int_C \Psi_\pm(z,\theta)\,\rho(\theta)\,d\theta\,,$$

where C is an integration contour and $\rho(\theta)$ is an appropriately chosen weight function. There will be three types of contours that will be important to us (see also Fig. 1):

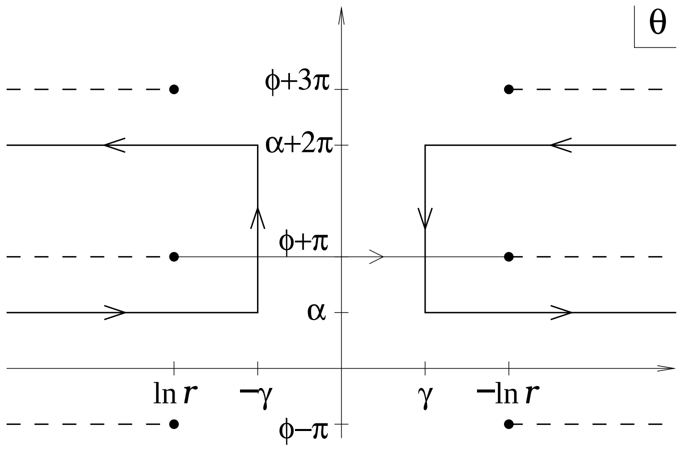

Figure 1: Contours of integration in the θ-plane

- Contour $C_+(z)$ starts at $-\infty + i\alpha$, surrounds the branch cut $\mathbf{b}_+ = \big(-\infty + i(\varphi + \pi), \ln r + i(\varphi+\pi)\big]$ in a counter-clockwise manner, and goes to $-\infty + i(\alpha+2\pi)$.
- Contour $C_-(z)$ starts at $\infty + i(\alpha+2\pi)$, then goes counter-clockwise around the branch cut $\mathbf{b}_- = \big[-\ln r + i(\varphi+\pi), \infty + i(\varphi+\pi)\big)$, and finally travels to $\infty + i\alpha$ along the ray parallel to the real axis.
- Contour $C_0(z)$ joins two branchpoints: $\theta_1 = \ln r + i(\varphi+\pi)$ and $\theta_2 = -\ln r + i(\varphi+\pi)$.

Real parameters α and γ can be chosen arbitrarily; the only conditions they should satisfy are given by

$$|\varphi - \alpha| < \pi, \qquad 0 \leq \gamma < -\ln r.$$

Assuming that $\operatorname{Re} k^2 < 0$, one may now write a number of contour integral representations for the radial waves (4.2)–(4.5):

$$\begin{aligned}
\mathrm{w}_l^{(I)}(z) &= \int_{C_0(z)} \Psi_-(z,\theta)\, e^{l\theta}\, d\theta, && (4.10)\\
\hat{\mathrm{w}}_l^{(I)}(z) &= \int_{C_0(z)} \hat{\Psi}_-(z,\theta)\, e^{-l\theta}\, d\theta, && (4.11)\\
\mathrm{w}_l^{(II,\pm)}(z) &= \pm\int_{C_\pm(z)} \Psi_+(z,\theta)\, e^{l\theta}\, d\theta, && (4.12)\\
\hat{\mathrm{w}}_l^{(II,\pm)}(z) &= \mp\int_{C_\mp(z)} \hat{\Psi}_+(z,\theta)\, e^{-l\theta}\, d\theta, && (4.13)
\end{aligned}$$

where the functions $\hat{\Psi}_\pm(z,\theta)$ are obtained from $\Psi_\pm(z,\theta)$ by replacing $b \to -b$. Although the validity of the representations (4.10)–(4.13) can be checked directly, their general structure may also be a posteriori understood as follows. Consider, for instance, the functions $\mathrm{w}_l^{(I)}(z)$ and $\mathrm{w}_l^{(II,\pm)}(z)$ as defined by (4.10) and (4.12). Continuation of these

functions along a counter-clockwise circuit enclosing the point $z=0$ amounts to simultaneous shift of the branch cuts and integration contours upwards by 2π in the θ-plane. This shift is in turn equivalent to simple multiplication of both functions by $e^{2\pi i l}$. Moreover, elementary solutions (4.9) satisfy the relation

$$\hat{L}\Psi_{\pm}(z,\theta) = (z\partial_z - \bar{z}\partial_{\bar{z}})\Psi_{\pm}(z,\theta) = -\partial_\theta \Psi_{\pm}(z,\theta),$$

which means that RHSs of (4.10) and (4.12) are common (multivalued) eigenfunctions of $\hat{H}^{(B)}$ and $\hat{L}$, their angular momenta being equal to l. First function is regular for $t \to 1$, since in this case the branch cuts pinch the imaginary axis. Similarly, the second function is regular for $t \to 0$. This implies (modulo constant factors that have to be found by a direct calculation) the relations (4.2) and (4.4).

4.2 Summation

Let us now turn to the calculation of the sums (4.7)–(4.8). For simplicity the case $|z| > |z'|$ is treated in detail and we only indicate the changes needed to handle another case. Substituting contour representations (4.2) and (4.5) into the relation (4.7), one obtains

$$\mathcal{G}_k^{(\pm)}(z,z') = \mp \sum_{l\in\mathbb{Z}+\nu,\, l \gtreqless 0} \int_{C_0(z)} d\theta_1 \int_{C_\mp(z')} d\theta_2 \; \Psi_-(z,\theta_1)\hat{\Psi}_+(z',\theta_2)\, e^{l(\theta_1-\theta_2)}.$$

Since $|z| > |z'|$, one may choose the contours $C_\pm(z')$ in such a way that $\gamma_{z'} > -\ln r$. Consequently, we have $\mathrm{Re}\,(\theta_1-\theta_2) < 0$ for all $\theta_1 \in C_0(z)$, $\theta_2 \in C_-(z')$ and $\mathrm{Re}\,(\theta_1-\theta_2) > 0$ for all $\theta_1 \in C_0(z)$, $\theta_2 \in C_+(z')$. Then it becomes possible to perform the summation inside the integrals and one finds

$$\mathcal{G}_k^{(+)}(z,z') + \mathcal{G}_k^{(-)}(z,z') = \int\limits_{C_0(z)} d\theta_1 \int\limits_{C_+(z')\cup C_-(z')} d\theta_2 \quad \Psi_-(z,\theta_1)\hat{\Psi}_+(z',\theta_2)\, \frac{e^{(1+\nu)(\theta_1-\theta_2)}}{e^{\theta_1-\theta_2}-1}$$

We would like to deform the contours $C_\pm(z')$ in the last integral over θ_2 so that their vertical parts compensate one another. Then $C_+(z')\cup C_-(z')$ transforms into two horizontal lines, but one also earns a pole contribution coming from $e^{\theta_2} = e^{\theta_1}$. Next, if we assume that $\varphi - \varphi' \neq \pm\pi$, then the two lines can be deformed into $\mathrm{Im}\,\theta_2 = \varphi'$ using quasiperiodicity in θ_2. Together with (4.6), this leads to the following representation for the Green function:

$$G_k(z,z') = \begin{cases} e^{-i\nu(\varphi-\varphi'+2\pi)}\, G_k^{(0)}(z,z') + \Delta_k(z,z') & \text{for } \varphi-\varphi' \in (-2\pi,-\pi), \\ e^{-i\nu(\varphi-\varphi')}\, G_k^{(0)}(z,z') + \Delta_k(z,z') & \text{for } \varphi-\varphi' \in (-\pi,\pi), \\ e^{-i\nu(\varphi-\varphi'-2\pi)}\, G_k^{(0)}(z,z') + \Delta_k(z,z') & \text{for } \varphi-\varphi' \in (\pi,2\pi)\,, \end{cases} \tag{4.14}$$

with

$$G_k^{(0)}(z,z') = \frac{1}{4\pi}\int_{C_0(z)} d\theta \;\; \Psi_-(z,\theta)\hat{\Psi}_+(z',\theta)\,, \tag{4.15}$$

$$\Delta_k(z,z') = \frac{1-e^{-2\pi i\nu}}{8i\pi^2}\, e^{-i\nu(\varphi-\varphi')} \int\limits_{C_0(z)} d\theta_1 \int\limits_{\operatorname{Im}\theta_2=\varphi'} d\theta_2 \;\; \Psi_-(z,\theta_1)\hat{\Psi}_+(z',\theta_2)\, \frac{e^{(1+\nu)(\theta_1-\theta_2)}}{e^{\theta_1-\theta_2}-1}\,. \tag{4.16}$$

Similarly, assuming that $|z|<|z'|$, one obtains an integral representation of the Green function which has exactly the same form as (4.14), except that the functions $G_k^{(0)}(z,z')$ and $\Delta_k(z,z')$ are now given by

$$G_k^{(0)}(z,z') = \frac{1}{4\pi}\int_{C_0(z')} d\theta \;\; \hat{\Psi}_-(z',\theta)\Psi_+(z,\theta)\,, \tag{4.17}$$

$$\Delta_k(z,z') = \frac{e^{2\pi i\nu}-1}{8i\pi^2}\, e^{-i\nu(\varphi-\varphi')} \int\limits_{C_0(z')} d\theta_1 \int\limits_{\operatorname{Im}\theta_2=\varphi} d\theta_2 \;\; \hat{\Psi}_-(z',\theta_1)\Psi_+(z,\theta_2)\, \frac{e^{(1+\nu)(\theta_2-\theta_1)}}{e^{\theta_2-\theta_1}-1}\,. \tag{4.18}$$

After some computations (technical details are outlined in the Appendix A) one may show that both representations coincide. Moreover, the integrals (4.15) and (4.17) can be carried out explicitly:

$$G_k^{(0)}(z,z') = \left(\frac{1-z\bar{z}'}{1-\bar{z}z'}\right)^b \zeta\Big(u(z,z')\Big), \tag{4.19}$$

where $u(z,z') = \left|\dfrac{z'-z}{1-\bar{z}z'}\right|^2$ has a simple relation with the geodesic distance between the points z and z', and the function $\zeta(u)$ is given by

$$\zeta(u) = \frac{1}{4\pi}\frac{\Gamma(\chi+b)\Gamma(\chi-b)}{\Gamma(2\chi)}\Big(1-u\Big)^\chi {}_2F_1\Big(\chi+b,\chi-b,2\chi,1-u\Big). \tag{4.20}$$

Note that $G_k^{(0)}(z,z')$ coincides with the well-known expression for the resolvent kernel of the Hamiltonian without AB field [9, 20]. This can also be seen directly from the representation (4.14), since $\Delta_k(z,z')$ in (4.16) or (4.18) obviously vanishes for $\nu=0$. The function $\Delta_k(z,z')$ may also be written in a symmetric form:

$$\Delta_k(z,z') = \frac{\sin\pi\nu}{\pi}\int\limits_{-\infty}^{\infty} d\theta\, \frac{e^{(1+\nu)\theta+i(\varphi-\varphi')}}{1+e^{\theta+i(\varphi-\varphi')}} \left(\frac{1+rr'e^{-\theta}}{1+rr'e^{\theta}}\right)^b \zeta\Big(v(r,r',\theta)\Big), \tag{4.21}$$

with

$$v(r,r',\theta) = \frac{r^2+r'^2+2rr'\cosh\theta}{1+r^2r'^2+2rr'\cosh\theta}\,. \tag{4.22}$$

In our opinion, the representation (4.14) and the formulas (4.19)–(4.22) constitute the most interesting results of the present paper. It is instructive to compare them with the known results in the flat space (cf. the relations (2.25)–(2.26) in [30] or the formula (5.10) from [27]). Notice that the 'free' part of the Green function is manifestly separated in (4.14) from the vortex-dependent contribution $\Delta_k(z,z')$.

5 Spectrum and density of states

The spectrum of the regular extension $\hat{H}_v^{\mathrm{reg}}$ consists of three parts:

- a continuous spectrum $E \in \left[(1+4b^2)/R^2\,,\infty\right)$;
- a finite number of infinitely degenerate eigenvalues, which coincide with the usual Landau levels on the hyperbolic disk [9, 24] in the absence of the AB field. These levels are explicitly given by

$$E_n^{(0)} = \frac{1}{R^2}\left[1+4b^2-4\left(|b|-n-\frac{1}{2}\right)^2\right], \tag{5.1}$$

  where $n = 0, 1, \ldots, n_{max} < |b| - 1/2$. Corresponding common eigenfunctions of the Hamiltonian $\hat{H}_v^{\mathrm{reg}}$ and the angular momentum operator $\hat{L}$ can be expressed in terms of Jacobi's polynomials (cf. the relation (13) in [24]):

$$\Psi_{n,l}^{(0)}(t,\varphi) \sim t^{|l+\nu|/2}(1-t)^{|b|-n} P_n^{(2|b|-2n-1,|l+\nu|)}(2t-1)\, e^{il\varphi}.$$

  Here one should take $l = 0, -1, -2, \ldots$ for $b > 0$ and $l = 1, 2, \ldots$ for $b < 0$.

- a finite number of bound states $E_n^{(\nu)}$ with finite degeneracy. The form of these eigenvalues depends on the sign of magnetic field. Namely, for $b > 0$ one has

$$E_n^{(\nu,+)} = \frac{1}{R^2}\left[1+4b^2-4\left(b-n-(1+\nu)-\frac{1}{2}\right)^2\right], \tag{5.2}$$

  where $n = 0, 1, \ldots, n'_{max} < b - (\nu+1) - 1/2$. In the case $b < 0$, the eigenvalues may be written as

$$E_n^{(\nu,-)} = \frac{1}{R^2}\left[1+4b^2-4\left(|b|-n+\nu-\frac{1}{2}\right)^2\right], \tag{5.3}$$

  with $n = 0, 1, \ldots, n''_{max} < |b| + \nu - 1/2$. Common eigenstates of $\hat{H}_v^{\mathrm{reg}}$ and $\hat{L}$ are again given by Jacobi's polynomials:

$$\begin{aligned} b>0: \quad & \Psi_{n,l}^{(\nu,+)}(t,\varphi) \sim t^{(l+\nu)/2}(1-t)^{b-n-(\nu+1)} P_n^{(2b-2n-2(\nu+1)-1,l+\nu)}(2t-1)\, e^{il\varphi}, \\ b<0: \quad & \Psi_{n,l}^{(\nu,-)}(t,\varphi) \sim t^{|l+\nu|/2}(1-t)^{|b|-n+\nu} P_n^{(2|b|-2n+2\nu-1,|l+\nu|)}(2t-1)\, e^{il\varphi}. \end{aligned}$$

  For given radial quantum number n the allowed eigenvalues of the angular momentum are $l = 1, 2, \ldots, n+1$ (for $b > 0$) and $l = 0, -1, \ldots, -n$ (for $b < 0$).

Remark. The above expressions (5.1)–(5.3) for the energy levels can also be extracted from the recent work [7]. It is worthwhile to emphasize that the discrete spectrum is absent for $|b| < 1/2$.

Let us now consider the density of states (DoS) on the hyperbolic disk. It can be obtained from the boundary values of the resolvent kernel on the real axis in the complex energy plane, using the following formula:

$$\rho(E) = \frac{1}{\pi}\,\mathrm{Im}\,\mathrm{Tr}\,G_k(z,z'\to z)\Big|_{k^2=E+i0}, \qquad E\in\mathbb{R}.$$

Both terms in the representation (4.14) of the Green function contribute to the DoS. The contribution of the free-resolvent kernel $G_k^{(0)}(z,z')$ has been first calculated by Comtet [9]. His results (supplemented by an additional term [6], coming from the discrete spectrum) give the following expression for the DoS:

$$\begin{aligned}\rho^{(0)}(E,z) &= \frac{1}{\pi}\,\mathrm{Im}\,G_k^{(0)}(z,z'\to z)\Big|_{k^2=E+i0} = \\ &= \frac{1}{4\pi}\frac{\sinh 2\pi\lambda}{\cosh 2\pi\lambda+\cos 2\pi b}\,\Theta\left(E-\frac{1+4b^2}{R^2}\right)+ \\ &\quad + \frac{2}{\pi R^2}\sum_{n=0}^{n_{max}}\left(|b|-n-\frac{1}{2}\right)\delta\left(E-E_n^{(0)}\right).\end{aligned}$$

Here $\Theta(x)$ denotes Heaviside function and

$$\lambda = \frac{1}{2}\sqrt{ER^2-1-4b^2}\,. \tag{5.4}$$

One can not expect that the DoS per unit area, induced by the AB field, will also be constant on D. However, it should depend only on the geodesic distance between a given point on the disk and the flux position. Indeed, since the function $\Delta(z,z')$ is non-singular for $z\to z'$, the vortex-dependent part of the DoS is given by

$$\rho^{(\nu)}(E,z) = \frac{1}{\pi}\,\mathrm{Im}\,\Delta_k(t)\Big|_{k^2=E+i0}, \tag{5.5}$$

where the function $\Delta_k(t)$ is obtained from $\Delta(z,z')$ by setting $\varphi=\varphi'$, $r^2=r'^2=t$:

$$\Delta_k(t) = \frac{\sin\pi\nu}{\pi}\int_{-\infty}^{\infty} d\theta\,\frac{e^{\,(1+\nu)\theta}}{1+e^{\,\theta}}\left(\frac{1+te^{-\theta}}{1+te^{\,\theta}}\right)^b \zeta\left(\frac{2t(1+\cosh\theta)}{1+t^2+2t\cosh\theta}\right). \tag{5.6}$$

As it stands, the representation (5.6) is valid in the left half-plane $\mathrm{Re}\,k^2<0$, where the function $\Delta_k(t)$ is analytic. However, the DoS is determined by the singularities of $\Delta_k(t)$ that occur on the positive part of the real axis (we may expect there a finite number of poles and the branch cut $\big[(1+4b^2)/R^2\,,\infty\big)$, corresponding to the continuous part of the spectrum of $\hat{H}_v^{\mathrm{reg}}$). One could try to construct the appropriate analytic continuation of $\Delta_k(t)$, considering (5.6) as a contour integral and then suitably deforming the contour. It seems, however, that this approach does not lead to any satisfactory result because of the complicated singularity structure of the function under the integral sign in the θ-plane.

An alternative method consists in the following. Remark that the vortex-dependent contribution to the DoS in the whole hyperbolic space

$$\rho^{(\nu)}(E) = \int_D d\mu \, \rho^{(\nu)}(E,z) \tag{5.7}$$

has a finite value, since

$$\left(\frac{1+te^{-\theta}}{1+te^{\,\theta}}\right)^b \zeta\left(\frac{2t(1+\cosh\theta)}{1+t^2+2t\cosh\theta}\right) =$$
$$= \begin{cases} -\dfrac{1}{4\pi}\Big[\ln 2t + \ln(1+\cosh\theta) + 2\gamma_E + \psi(\chi+b) + \psi(\chi-b)\Big] + o(1) & \text{for } t\to 0, \\ \dfrac{1}{4\pi}\dfrac{\Gamma(\chi+b)\Gamma(\chi-b)}{\Gamma(2\chi)}\dfrac{(1-t)^{2\chi}}{(1+e^{\theta})^{\chi+b}(1+e^{-\theta})^{\chi-b}} + o\left((1-t)^{2\chi}\right) & \text{for } t\to 1. \end{cases}$$

If one now integrates $\Delta_k(t)$ over spatial coordinates (see Appendix B) and then considers the analytic continuation of the result to the complex energy plane, the following expression for $\rho^{(\nu)}(E)$ can be obtained:

$$\begin{aligned} \rho^{(\nu)}(E) = -\frac{R^2}{4\pi}\,\mathrm{Im}\,\frac{1}{2\chi-1}\Big\{(\chi-b+\nu)\Big[\psi(\chi-b)-\psi(\chi-b+\nu+1)\Big]+ \\ + (\chi+b-\nu-1)\Big[\psi(\chi+b)-\psi(\chi+b-\nu-1)\Big]\Big\}\Big|_{k^2=E+i0} = \\ = \rho_d^{(\nu)}(E) + \rho_c^{(\nu)}(E), \end{aligned} \tag{5.8}$$

where the contributions of the discrete and continuous part of the spectrum are given by

$$\rho_d^{(\nu)}(E) = \begin{cases} \sum\limits_{n=0}^{n'_{max}} (n+1)\,\delta\left(E-E_n^{(\nu,+)}\right) - \sum\limits_{n=0}^{n_{max}} (n-\nu)\,\delta\left(E-E_n^{(0)}\right) & \text{for } b>0, \\ \sum\limits_{n=0}^{n''_{max}} (n+1)\,\delta\left(E-E_n^{(\nu,-)}\right) - \sum\limits_{n=0}^{n_{max}} (n+\nu+1)\,\delta\left(E-E_n^{(0)}\right) & \text{for } b<0, \end{cases} \tag{5.9}$$

$$\begin{aligned} \rho_c^{(\nu)}(E) = -\frac{R^2}{8\lambda}\,\Theta\left(E-\frac{1+4b^2}{R^2}\right)\Bigg\{\frac{\lambda\sinh 2\pi\lambda + \left(\frac12 - b+\nu\right)\sin 2\pi(b-\nu)}{\cosh 2\pi\lambda + \cos 2\pi(b-\nu)} - \\ - \frac{\lambda\sinh 2\pi\lambda + \left(\frac12 - b+\nu\right)\sin 2\pi b}{\cosh 2\pi\lambda + \cos 2\pi b}\Bigg\}, \end{aligned} \tag{5.10}$$

and the parameter λ is defined as in (5.4).

At last we add a comment concerning the flat space limit ($R\to\infty$) at zero magnetic field ($b=0$). In this case the representation (5.8) for the vortex-dependent DoS transforms into

$$\rho^{(\nu)}(E) \overset{R\to\infty}{\longrightarrow} \frac{1}{\pi}\,\mathrm{Im}\,\frac{\nu(\nu+1)}{2k^2}\Big|_{k^2=E+i0} = -\frac{\nu(\nu+1)}{2}\,\delta(E)\,. \tag{5.11}$$

This result has been first obtained in [10], and it has important consequences in the theory of disordered magnetic systems (see, for example, [13, 14, 15]). Obtaining the relation (5.11) directly from (5.10) is more subtle; one should consider $\rho_c^{(\nu)}(E)$ as a distribution and supply it with a proper regularization at the edge of the spectrum, i. e. as $\lambda \to +0$.

6 Discussion

We have studied the Hamiltonian of a particle moving on the hyperbolic disk in the background of a uniform magnetic field and the AB gauge potential. The density of states and the resolvent of this operator have been calculated in a closed form, using Sommerfeld-type integral representations for the radial waves.

The above discusssion does not exhaust all problems related to the system under consideration. First of all, there remains a technical question of analytic continuation of the representation (4.21) to energy values with $\operatorname{Re} k^2 \geq 0$. Such continuation would allow to investigate the curvature dependence of various vacuum quantum numbers, e. g. fractional charge, magnetic flux and angular momentum (see, for instance, [33]), induced by the AB vortex. One may also try to address the latter problem, applying the technique we have used in the calculation of the density of states.

Another interesting question concerns the generalization of our results to the case of the Dirac Hamiltonian. This problem, in its turn, appears to be non-trivially related to the theory of isomonodromic deformations on the hyperbolic disk [28, 31, 32]. More precisely, it was shown in [31] that the two-point isomonodromic tau function of the Dirac operator on the Poincaré disk provides a solution to Painlevé VI equation. The explicit form of this Painlevé transcendent in a particular case of zero magnetic field has been conjectured by Doyon [17], whose idea was to replace the tau function by its physical analog — a correlator of monodromy fields, and to sum up the corresponding form factor expansion. The missing ingredient in the rigorous proof of this result and its generalization to the case of non-zero field is the Green function of the Dirac operator with one branch point, precisely analogous to the resolvent found in the present paper. Moreover, the knowledge of this Green function enables one to compute all form factors of monodromy fields on the hyperbolic disk [17], in particular, those of Ising spin and disorder fields [18, 19]. We leave these problems to a future publication.

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Appendix A

Here we describe a method of obtaining the expressions (4.19)–(4.22) for the Green function from the integral representations (4.15)–(4.18). First, note that horocyclic waves

(4.9) satisfy the following identity:

$$\Psi_\pm(z,\theta) = \left(\bar\alpha + \beta e^{-\theta}\right)^{-\chi_\pm + b}\left(\alpha + \bar\beta e^{\theta}\right)^{-\chi_\pm - b}\left(\frac{\beta\bar s + \alpha}{\bar\beta s + \bar\alpha}\right)^b \Psi_\pm(s,\xi), \tag{A.1}$$

where we have introduced the notation

$$s = \frac{\bar\alpha z - \beta}{-\bar\beta z + \alpha}, \qquad e^{\xi} = \frac{\bar\alpha\, e^{\theta} + \beta}{\bar\beta\, e^{\theta} + \alpha} \tag{A.2}$$

and assumed that $|\alpha|^2 + |\beta|^2 = 1$. Let us now make the change of variables (A.2) in (4.15) (introducing similarly the variable $s' = \dfrac{\bar\alpha z' - \beta}{-\bar\beta z' + \alpha}$ instead of z'). It is then straightforward to check that (4.15) reduces to

$$G^{(0)}_k(z,z') = \frac{1}{4\pi}\left(\frac{\beta\bar s + \alpha}{\bar\beta s + \bar\alpha}\right)^b \left(\frac{\bar\beta s' + \bar\alpha}{\beta\bar s' + \alpha}\right)^b \int_{C_0(s)} d\xi\;\; \Psi_-(s,\xi)\hat\Psi_+(s',\xi) \tag{A.3}$$

We may now adjust the parameters α and β in such a way that $s' = 0$. For example, one can take

$$\alpha = \frac{1}{\sqrt{1-|z'|^2}}\,, \qquad \beta = \frac{z'}{\sqrt{1-|z'|^2}}\,.$$

Since in this case $s = \dfrac{z - z'}{1 - z\bar z'}$, the relation (A.3) transforms into

$$G^{(0)}_k(z,z') = \frac{1}{4\pi}\left(\frac{1 - z\bar z'}{1 - \bar z z'}\right)^b \int_{C_0(s)} d\xi\;\; \Psi_-(s,\xi) =$$

$$= \frac{1}{4\pi}\left(\frac{1 - z\bar z'}{1 - \bar z z'}\right)^b \left(1 - |s|^2\right)^{\chi_-} \int\limits_{\ln|s|}^{-\ln|s|} \frac{d\xi}{(1 - |s| e^{-\xi})^{\chi_- - b}\,(1 - |s| e^{\xi})^{\chi_- + b}}\,. \tag{A.4}$$

After the change of variable $e^{\xi} = \dfrac{1 - (1 - |s|^2)t}{|s|}$ in the last integral, we obtain standard representation of Gauss hypergeometric function,

$$G^{(0)}_k(z,z') = \frac{1}{4\pi}\left(\frac{1 - z\bar z'}{1 - \bar z z'}\right)^b \left(1 - |s|^2\right)^{\chi} \int\limits_0^1 \frac{t^{\chi - b - 1}(1-t)^{\chi + b - 1} dt}{\left(1 - (1 - |s|^2)\, t\right)^{\chi + b}}\,,$$

which immediately gives (4.19)–(4.20). Performing analogous manipulations with the representation (4.17), one finds the same answer.

Let us now consider the representation (4.16) for the function $\Delta_k(z,z')$. Interchanging the order of integration and then introducing instead of θ_2 a new variable $\tilde\theta_2 = \theta_2 - \theta_1$, one can check that this relation transforms into

$$\Delta_k(z,z') = \frac{1 - e^{-2\pi i\nu}}{2\pi i}\, e^{-i\nu(\varphi - \varphi')} \int\limits_{\operatorname{Im}\tilde\theta_2 = \varphi' - \varphi - \pi} d\tilde\theta_2\, \frac{e^{-(1+\nu)\tilde\theta_2}}{e^{-\tilde\theta_2} - 1}\, F(z,z',\tilde\theta_2)\,, \tag{A.5}$$

where the function $F(z,z',\tilde{\theta}_2)$ after additional change of integration variable $\theta_1 \to \tilde{\theta}_1 = \theta_1 + \tilde{\theta}_2$ can be written as

$$F(z,z',\tilde{\theta}_2) = \frac{1}{4\pi}\int_{C_{\tilde{\theta}_2}(z)} d\tilde{\theta}_1\ \Psi_-(z,\tilde{\theta}_1-\tilde{\theta}_2)\hat{\Psi}_+(z',\tilde{\theta}_1)\,. \tag{A.6}$$

In the last expression, $C_{\tilde{\theta}_2}(z)$ represents the integration contour obtained from $C_0(z)$ by shifting it by $\tilde{\theta}_2$.

One may now use the trick described above to eliminate the function $\hat{\Psi}_+(z',\tilde{\theta}_1)$ from (A.6). Namely, consider further change of variables:

$$\tilde{\theta}_1 \to \xi, \qquad e^{\,\xi} = \frac{e^{\,\tilde{\theta}_1}+z'}{\bar{z}'e^{\,\tilde{\theta}_1}+1}\,.$$

Somewhat tedious but fairly routine calculation shows that the integral (A.6), being rewritten in terms of ξ, reduces to

$$F(z,z',\tilde{\theta}_2) = \frac{1}{4\pi}\left(\frac{1+z'\tilde{s}}{1+\bar{z}'s}\right)^b (1-s\tilde{s})^{\chi_-}\int_{C_\xi}\frac{d\xi}{(1+s\,e^{-\xi})^{\chi_--b}(1+\tilde{s}\,e^{\,\xi})^{\chi_-+b}}\,, \tag{A.7}$$

where we have introduced the notation

$$s = -\frac{r\,e^{\,\mathrm{Re}\,\tilde{\theta}_2}+r'}{1+rr'\,e^{\,\mathrm{Re}\,\tilde{\theta}_2}}\;e^{i\varphi'}, \qquad \tilde{s} = -\frac{r\,e^{-\mathrm{Re}\,\tilde{\theta}_2}+r'}{1+rr'\,e^{-\mathrm{Re}\,\tilde{\theta}_2}}\;e^{-i\varphi'}. \tag{A.8}$$

Contour C_ξ denotes the horizontal line segment in the ξ-plane, joining the points $\ln|s|+i\varphi'$ and $-\ln|\tilde{s}|+i\varphi'$. Rewriting (A.7) in terms of ordinary integrals and using (A.8), one gets a relation analogous to (A.4):

$$\begin{aligned} F(z,z',\tilde{\theta}_2) \;&=\; \frac{1}{4\pi}\left(\frac{1+rr'\,e^{\,\mathrm{Re}\,\tilde{\theta}_2}}{1+rr'\,e^{-\mathrm{Re}\,\tilde{\theta}_2}}\right)^b (1-s\tilde{s})^{\chi_-}\,\times \\ &\quad\times\int\limits_{\frac{1}{2}\ln|s\tilde{s}|}^{-\frac{1}{2}\ln|s\tilde{s}|}\frac{d\xi}{\left(1-|s\tilde{s}|^{1/2}\,e^{-\xi}\right)^{\chi_--b}\left(1-|s\tilde{s}|^{1/2}\,e^{\,\xi}\right)^{\chi_-+b}} = \\ &=\;\left(\frac{1+rr'\,e^{\,\mathrm{Re}\,\tilde{\theta}_2}}{1+rr'\,e^{-\mathrm{Re}\,\tilde{\theta}_2}}\right)^b \zeta\Big(|s\tilde{s}|\Big). \end{aligned} \tag{A.9}$$

Finally, combining the last formula with (A.5) and (A.8), we find the relations (4.21)–(4.22). The same result can be obtained from the representation (4.18) in a completely analogous manner.

Appendix B

In order to integrate the function $\Delta_k(t)$ over the hyperbolic disk, consider the following identity:

$$\begin{aligned}
&\left(\frac{1+te^{-\theta}}{1+te^{\,\theta}}\right)^b \zeta\left(\frac{2t(1+\cosh\theta)}{1+t^2+2t\cosh\theta}\right) = \\
&= \frac{(1-t)^{2\chi}}{4\pi}\int\limits_{-\infty}^{\infty}\frac{d\theta'}{[(1+e^{-\theta})(1+e^{-\theta'})-(1-t)e^{-\theta}]^{\chi-b}}\times \\
&\times \frac{1}{[(1+e^{\theta})(1+e^{\theta'})-(1-t)(e^{\theta}+e^{\theta'}+e^{\theta+\theta'})]^{\chi+b}}\,.
\end{aligned}$$

Substituting this relation into the integral $\int_D d\mu\, \Delta_k(t)$, integrating over the angle φ and introducing instead of t a new variable $s=1-t$, one obtains

$$\int_D d\mu\, \Delta_k(t) = R^2\,\frac{\sin\pi\nu}{4\pi}\int_{-\infty}^{\infty}d\theta\int_{-\infty}^{\infty}d\theta'\,\frac{e^{\,(1+\nu)\theta}}{1+e^{\,\theta}}\,\mathcal{F}(\theta,\theta'), \tag{B.1}$$

$$\mathcal{F}(\theta,\theta') = \int_0^1 \frac{s^{2\chi-2}\,ds}{[(1+e^{-\theta})(1+e^{-\theta'})-se^{-\theta}]^{\chi-b}\,[(1+e^{\theta})(1+e^{\theta'})-s(e^{\theta}+e^{\theta'}+e^{\theta+\theta'})]^{\chi+b}}\,.$$

After further change of variable $s\to u=\dfrac{s}{s+(1+e^{\theta})(1+e^{\theta'})(1-s)}$ the function $\mathcal{F}(\theta,\theta')$ can be rewritten as

$$\mathcal{F}(\theta,\theta') = \frac{1}{(1+e^{\theta})(1+e^{\theta'})}\int_0^1\frac{u^{2\chi-2}du}{[e^{-\theta-\theta'}+(1+e^{-\theta'})u]^{\chi-b}}\,.$$

Integrating once by parts, one finds

$$\begin{aligned}
\mathcal{F}(\theta,\theta') \;&=\; \frac{\left[1+e^{-\theta'}+e^{-\theta-\theta'}\right]^{-(\chi-b)}}{(2\chi-1)(1+e^{\theta})(1+e^{\theta'})}+ && \text{(B.2)}\\
&\quad+\frac{(\chi-b)e^{-\theta'}}{(2\chi-1)(1+e^{\theta})}\int_0^1\frac{u^{2\chi-1}du}{[e^{-\theta-\theta'}+(1+e^{-\theta'})u]^{\chi-b+1}}\,. && \text{(B.3)}
\end{aligned}$$

Let us consider the contribution of the term (B.2) to the integral (B.1). Introducing instead of θ' a new variable $v=[1+e^{-\theta'}+e^{-\theta-\theta'}]^{-1}$, we may write

$$\int_{-\infty}^{\infty}d\theta\int_{-\infty}^{\infty}d\theta'\,\frac{e^{\,(1+\nu)\theta}}{1+e^{\,\theta}}\,\mathcal{F}_1(\theta,\theta') = \frac{1}{2\chi-1}\int_0^1 dv\int_{-\infty}^{\infty}d\theta\,\frac{e^{\,(1+\nu)\theta}}{(1+e^{\,\theta})^2}\,\frac{v^{\chi-b-1}}{1+ve^{-\theta}}\,. \tag{B.4}$$

In the contribution of the term (B.3), we first perform the integration over θ', then replace $\theta\to-\theta$ and finally exchange the order of integration over θ and u. The result looks quite similar to (B.4):

$$\int_{-\infty}^{\infty}d\theta\int_{-\infty}^{\infty}d\theta'\,\frac{e^{\,(1+\nu)\theta}}{1+e^{\,\theta}}\,\mathcal{F}_2(\theta,\theta') = \frac{1}{2\chi-1}\int_0^1 du\int_{-\infty}^{\infty}d\theta\,\frac{e^{-\nu\theta}}{(1+e^{\,\theta})^2}\,\frac{u^{\chi+b-1}}{1+ue^{-\theta}}\,. \tag{B.5}$$

The integrals over θ in (B.4)–(B.5) can be easily calculated by residues. For example, one obtains

$$\frac{\sin \pi\nu}{\pi} \int_{-\infty}^{\infty} d\theta \, \frac{e^{(1+\nu)\theta}}{(1+e^{\theta})^2(1+ve^{-\theta})} = \frac{v^{1+\nu} - 1 + (1+\nu)(1-v)}{(1-v)^2} \, .$$

Subsequent integration over v and u leads to the final result:

$$\begin{aligned} \int_D d\mu \, \Delta_k(t) \quad &= \quad -\frac{R^2}{4(2\chi-1)} \, \Big\{ (\chi - b + \nu) \Big[\psi(\chi - b) - \psi(\chi - b + \nu + 1) \Big] + \\ & \qquad + (\chi + b - \nu - 1) \Big[\psi(\chi + b) - \psi(\chi + b - 1 - \nu) \Big] \Big\} \, , \end{aligned}$$

where $\psi(x)$ denotes the digamma function.

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